ELECTION LAW JOURNAL
Volume 15, Number 4, 2016

DOI: 10.1089/elj.2016.0384

Toward a Talismanic Redistricting Tool: A Computational Method for Identifying Extreme Redistricting Plans

Wendy K. Tam Cho and Yan Y. Liu

ABSTRACT

Partisan gerrymandering is widely frowned upon by the citizenry as well as the Supreme Court. Despite broad disdain for the practice, the Supreme Court has found it difficult to identify a workable standard by which we might regulate political gerrymandering. We have lacked sufficient tools to analyze and synthesize redistricting data, in part, because the requisite computation is massive. At the same time, the recent proliferation of significant computing power has led to the discovery of the extensive and often surprising reach of technology, information, and computation in many realms of life. Our capacities to compile, organize, analyze, and disseminate information have increased dramatically and facilitated the creation of many tools to connect citizens and automate human tasks. We present a computational model that brings these significantly advanced computing capacities to the redistricting process. Our model allows us to understand redistricting in fundamentally new ways and allows us to integrate technological advances with our articulated theories for redistricting and democratic rule while also empowering citizens with new abilities to understand and overturn partisan gerrymanders.

Keywords: redistricting, supercomputing, partisan gerrymandering

THE SEARCH FOR A STANDARD

In the five decades since *Baker v. Carr*,[1] neither the Supreme Court nor the election law community has been able to settle on a standard for, or a definition of, political fairness. Moreover, it is unclear how this "fairness" could be balanced with other criteria or implemented as a practical matter by the many entities that draw district lines in the United States. In *Vieth v. Jubelirer*,[2] the Court commented that developing such a standard might be so sufficiently difficult that partisan gerrymandering may fall into the realm of non-justiciability. The Court, however, did not go quite this far and left open the idea of an eventual workable standard. *LULAC v. Perry*[3] revived this idea under the rubric of partisan symmetry. In the words of the Court, a measure of partisan symmetry may be "a helpful (though certainly not talismanic) tool." In the last few decades, many different measures and perspectives have been proposed, but one has yet to satisfy the Court. This failure has not been for lack of effort.

Early on, some hoped that formal criteria like compactness, contiguity, and very strict equal

Wendy K. Tam Cho is a professor in the Department of Political Science and the Department of Statistics and a senior research scientist at the National Center for Supercomputing Applications at the University of Illinois at Urbana-Champaign. Yan Y. Liu is a senior research programmer at the Cyberinfrastructure and Geospatial Information Laboratory, Department of Geography and Geographic Information Science, and the National Center for Supercomputing Applications at the University of Illinois at Urbana-Champaign.

[1] 369 U.S. 186 (1962).
[2] 541 U.S. 267 (2004).
[3] 548 U.S. 399 (2006).

population rules would prevent legislators and their consultants from drawing politically unfair lines (Schwartzberg 1965; Grofman 1985; Polsby and Popper 1991; Niemi et al. 1990). However, purely formalistic approaches limited only some kinds of political gerrymanders, notably those that most egregiously employed non-compact shapes to either "pack" or "crack" opponents. Other types of gerrymanders that disproportionately packed opponents in the name of compactness continued unabashed (Cain 1985; Lowenstein and Steinberg 1985). There has been obvious success and even surprising ease with evading the legal constraints that are intended to limit unfair gerrymandering.

The failure of the legal system in this political realm has become more poignant alongside the recent rise of partisan and bipartisan redistricting cases in the state and federal courts. This increasing litigation coupled with rising frustration has fueled a renewed desire to develop appropriate checks on the redistricting process. The legal challenges have made it clear that judges continue to lack a standard by which to rule on the constitutionality of disputed redistricting plans. A measure of partisan gerrymandering is highly desirable but elusive.

A NEW FRAMEWORK: FIRST AMENDMENT VERSUS FOURTEENTH AMENDMENT

Surely, the guidance and basis for constructing a measure must be rooted in the framework in which it will be utilized. Recently the courts have expanded their view in this regard, providing more information and a potentially more expansive legal basis upon which we might build a partisan gerrymandering standard. In *Shapiro v. McManus*,[4] the Supreme Court opened a new possible avenue for challenging partisan gerrymanders via the First Amendment. Similarly, in *Whitford v. Nichol*,[5] a lower court heard a partisan gerrymandering case in Wisconsin where plaintiffs allege that the Wisconsin Republicans violated the First and Fourteenth amendments by drawing state assembly districts that produced a nearly 10% efficiency gap bias in 2010 and a 13% efficiency gap in 2012.

In a Fourteenth Amendment equal protection claim, the core question is whether a political party or racial group has been treated unequally by the redistricting map in question. Under the First Amendment framework, the question shifts to whether a state action burdens the exercise of speech in some meaningful way based on speech content as revealed by past voting behavior and registration. Any entity that utilizes voting data from partisan races is likely conditioning state action on speech. While the Court recognizes that a redistricting plan might need to treat political parties differently in order to achieve other important state goals, these infringements should be limited and not excessive. In incumbent or bipartisanship gerrymanders that deny voters the chance to use their votes to effect change in legislative representation, one might argue that jurisdictions that use political data in redistricting are conditioning state action (i.e., district design) on the content of past speech (e.g., previous vote history or voter registration) in order to create safe incumbent seats or safe Democratic or Republican held seats.

Notably, the structure and types of arguments that can be pursued expand with an extension to the First Amendment. The key to either an equal protection or First Amendment claim, however, is the ability to quantify the degree to which partisanship factors into the creation of a map. Since the use of partisan data is allowable as long as it is not excessive, one then needs to have a measure of how much partisanship, versus other criteria, has factored into the process. An analytical method needs to be able to separate natural consequences arising from particular population concentrations from state imposed disparate effects that bestow an unnecessary political advantage in favor of one group over another. Such a tool is not readily available, making it currently difficult to determine when a partisan redistricting has surpassed legal notions of fairness.

MEASUREMENT AND TOOLS

It has been clear for some time that the Supreme Court will consistently reject proportional representation as a constitutionally mandated fairness standard for either racial or partisan gerrymandering (see *Davis v. Bandemer*[6] and *Johnson v. De Grandy*[7]). Various other proposals have been made, but not

---

[4] 577 U.S. ___ (2015).
[5] Case 3:15-cv-00421-bbc (W.D. Wisc. 2016).
[6] 478 U.S. 109 (1986).
[7] 512 U.S. 997 (1994).

accepted by the Court. For instance, some have urged the courts to adopt a seats–votes ratio measure that would be more compatible with the American single member, simple plurality district election rules (Niemi and Deegan 1978; Niemi 1985; Niemi and Jackman 1991). Others have suggested that the Court adopt a symmetry standard that, in its most idealized form, would yield an identical share of seats for any given share of the votes (Grofman and King 2007). Yet another measure might arise from computing the "efficiency gap," which seeks to capture the difference in wasted votes between two parties in an election (McGhee 2014; Stephanopoulos and McGhee 2015).

Even if a measure could be agreed upon, the necessary tools need to accompany these measures. After all, a measure is only as good as the tool that provides the measurement. Consider that if one were interested in determining the length of a black garden ant in millimeters, someone might propose that a measurement tool, like a ruler, would not only be useful but would be the correct tool. However, if a person then produced a ruler that only had marks for inches on it, the measurements would not be very good or particularly precise despite the great value and correctness embodied in the *idea* of a ruler. Because of this shortcoming in the tool, one should plainly understand the announced length of the black garden ant with the knowledge that the ruler had only marks for inches while black garden arts are less than an inch long. Similarly, one can create an even less precise tool, perhaps a stick with no marks on it at all. Again, the idea for the measurement is the same and remains theoretically sound, but the empirical results clearly need to be interpreted and judged by the quality or crudeness of the tool.

In redistricting, various measurements and tools have been proposed. A theoretically attractive approach is to examine a plan in relationship to the full enumeration of possible plans. With a full enumeration, one can determine how any particular plan compares on any facet. With the full distribution, one can see if a plan is particularly biased toward one party or the other, or whether it has less or more respect for political subdivisions than other plans with comparable metrics. One could compare against the whole slew of plans or with perhaps only those plans that surpass some level of respect for political subdivisions. Indeed, any question of this type can be answered. As attractive as this approach is, the tools to create a full enumeration elude us (Vickrey 1961; Nagel 1965; Weaver and Hess 1963; Harris 1964). The computational complexity of the problem renders this theoretically attractive approach infeasible in the foreseeable computing environment. Indeed, all efforts in this direction have been successful for only *very* small redistricting applications (at high levels of granularity) that bear scant resemblance to actual redistricting problems (Garfinkel and Nemhauser 1970; Gudgin and Taylor 1979; Papayanopoulos 1973; Shepherd and Jenkins 1970).

If appropriate and useful tools for these theoretically attractive measurements did exist, the Court would be able to engage in productive discussions of how they might be utilized. Unfortunately, the extant tools are crude. This is not for lack of direction or desire. Instead, the required computational power is insufficient to realize the theoretic constructs. Attempts thus far have made a slew of substantively unrealistic simplifications that ignore critical redistricting requirements such as the Voting Rights Act and traditional districting principles such as preserving cities, political subdivisions, and other communities of interest, elements that are often mandated by either the Supreme Court, state constitutions, or local government charters, in order to gain computational tractability. From a legal vantage point, then, these simplistic and crude tools are useful only for addressing a different substantive problem that does not bear sufficient resemblance to the reality of political redistricting.

ESTABLISHING A BASELINE

For any tool that assesses the extent of partisan gerrymandering, one needs a *baseline* for comparison. To be sure, we are not the first to propose that a baseline is necessary. The principle that a baseline needs to be established has widespread acceptance and harkens back to discussions five decades ago in the literature about full enumeration approaches. Quite clearly, the purpose of the full enumeration is to establish a baseline. It is reasonably simple to measure any number of plan characteristics, including the seats–votes ratio, symmetry measures, efficiency measures, etc. Any plan has a particular level of competitiveness, a specific efficiency, a level of responsiveness to changing vote proportions, etc., but how do we understand these numbers that are produced? What does it mean that a plan is associated with a particular set of numbers that represent measures? Are these numbers extreme in the given jurisdiction? Obviously, to understand the measurements for any one plan, we must view them in the proper context. How competitive the districts *could* have been plainly goes a long way in helping us to understanding a single competitiveness number because it provides information on the range of values for other possible plans. In short, we need the context of a *large number* of possible plans to understand the characteristics of any single plan.

A critical component in establishing any baseline is that *the population under consideration must remain constant*. If the population is different for different plans, then it is not possible to determine whether the differences in the measure are truly differences in the measure or simply artifacts from altering the underlying data. For example, Stephanopoulos and McGhee (2015) compute their efficiency gap for congressional and state houses between 1972 and 2012. Over this time period, they found that redistricting plans were fairly balanced, offering neither party a significant advantage over the other. However, they also note that redistricting plans have exhibited an increasing efficiency gap over time in favor of Republicans and that the current districting plans are the most biased in our modern history. Certainly, the change in magnitude and direction of the efficiency gap across time and elections is interesting and intriguing. It is possible that the phenomenon is attributable to wide-scale changes that favor Republicans in the redistricting process. However, simply noting an increasing efficiency gap over time in favor of Republicans would not be sufficient for inferring a conclusion about the changing and increased biasedness of redistricting plans because this scenario is also consistent with the phenomenon of geographic sorting, whereby partisan neighborhoods become increasingly homogeneous through migration patterns (Bishop 2008). Via population migration, even if electoral maps were fixed, the efficiency gap would still be affected by changes in the underlying population.

To be sure, since the fundamental impetus for redistricting is to allay the effects from population shifts in the ten-year interim, we are mindful that the geographic distribution of voters is fluid. Accordingly, comparisons to metrics from other plans must include only plans with the same population base. Data from the same state but a different year will not include the same voter base. Data from other states is problematic because the population differs and those plans may not seek to adhere to the same redistricting criteria. The population across comparison plans must be constant.

In addition to creating a set of possible maps where the underlying population is constant, the maps in the baseline comparison set need to be a *fully balanced comparison set* of plans, meaning that they balance and consider the full set of relevant redistricting criteria. The full set of criteria may vary by locale. Different state and local entities have the power to impose different criteria. Usually, there is a core set of criteria that includes population equality, contiguity, and constraints on compactness and preserving communities of interest, cities, and counties. Of course, it is an uncontroversial claim that the considered maps satisfy all legal criteria. No one would claim otherwise. However, in practice, this is sometimes not followed because as more criteria are simultaneously considered, the computational problem becomes increasingly difficult and quickly unmanageable. On the other hand, when the full set of criteria are not considered, the produced comparison set is substantively less interesting and perhaps substantively uninteresting.

Finally, another criterion on top of fully balanced plans with a constant underlying population is that we need to have a *large set of independent maps*. Plainly, comparing to just one other plan is problematic. It may seem that 100 or 1,000 plans would be much preferred. However, if these 1,000 plans are chosen in such a way that they are highly similar, then having 1,000 plans is only marginally better than having a single comparison plan. The maps should be unique from one another, each contributing new information, so that we are able to gain some understanding of what is possible in map-making for a particular jurisdiction.

THE REALM OF POSSIBILITY

In the world of redistricting, there is not a perfect fit between what is possible and what is ideal. Ideally, we would like a full enumeration of possible plans. Such a set of plans allows us to fully understand any one plan in context, thus allowing us to draw conclusions about whether the plan is biased toward one party or the other, or to determine whether other plans might more satisfactorily fulfill the pre-stated requirements and desires of various stakeholders. Unfortunately, computational complexity precludes the possibility of a full enumeration. When the full set is astronomically large, it is also not straightforward to generate a sufficiently large, informative, and random draw from the full set.

We submit a novel, feasible, and promising strategy. Among the astronomically large set of possible plans, the key insight into making the exercise even remotely plausible is that the set of interesting, reasonable, and relevant plans is a much smaller subset. In the literature, this has been termed the set of "reasonably imperfect plans" (Cain 2012). These are plans that satisfy some set of goodness measures (on, for example, compactness, preserving political subdivisions, population equality, competitiveness, efficiency, responsiveness, biasedness, etc.) so that they are acceptable to relevant and important interest groups. The vast majority of "random plans" are not feasible or relevant in the sense that no one would ever consider enacting these plans. If the plan would not be a serious contender as an actual redistricting map, then it should not be in a comparison set with an enacted redistricting plan. Plans that fulfill only minimum legal requirements such as contiguity and population equality, but are not acceptable by the parties, citizen groups, racial minorities, or good government protectors, are not in the set of reasonably imperfect plans. The set of reasonably imperfect plans is defined by the peculiarities, laws, and actors in the jurisdiction under consideration.

This set is "imperfect" because there is no perfect plan that will satisfy all groups; there will always be tradeoffs because different groups likely have distinct, and possibly conflicting, interests and objectives. A reasonable plan is not simply a plan that fulfills minimal legal requirements. Moreover, being a "random plan" does not make a plan reasonable. A reasonable plan is a plan that a human might have created based on a set of criteria that actual line drawers consider and value. It is a plan that takes a variety of different interests into account and seeks to balance competing criteria. It will not perfectly satisfy any one group, but it will cross thresholds of goodness and desirability and be reasonably imperfect to many groups.

Simulation

One way to create a useful and theoretically attractive baseline comparison set of reasonably imperfect plans is via simulation. The idea of using computers to produce simulations for redistricting is not new. Indeed, nearly 50 years ago, Thoreson and Liittschwager (1967) proposed the idea and programmed their model into the University of Iowa's IBM 7044 digital computer. Even on computing technology from as far back as the 1960s, they were able to produce 150 simulated plans that satisfied measures of population equality, compactness, and contiguity. To be sure, they enacted simplifications to make the computation possible. Notably, they conducted their analysis at the level of counties, a level at which no actual redistricting is conducted. However, using smaller geographic units makes the problem prodigiously larger and unrealistic on their computing platform.

Fifty years later, Chen and Rodden (2015) proposed again that simulation is a useful approach. Their innovation cannot be in using simulation for redistricting analysis. That is an old idea. Instead, rather than proposing a new measurement idea, their contribution is updating the precision of the simulation tool. Instead of using a computer at the cutting edge in the 1960s, they had at their disposal the advances in computing technology over the past 50 years, which, as we know from Moore's Law, are significant (Moore 1965). Though the underlying theory is not different, they conduct a more fine-grained analysis. An enduring source of computational complexity is rooted in working with small units, which is why virtually all earlier attempts have been at the county level. Chen and Rodden claim to have 7,349 precincts or precinct clusters in their analysis, but they then go on to say that their "analytical solution is to grant deference to the state government and simply hold clusters of precincts—or even entire districts—fixed… " In their analysis of Florida, they hold three entire districts constant as well as leave 46 of the 67 Florida counties and 384 of Florida's 410 incorporated municipalities intact. Of the almost 7,000 precincts in Florida, if we assume that counties, districts, and cities are comprised roughly of the same number of precincts, after their aggregation into these large fixed units, there are far fewer units that can be manipulated. How many they actually work with is unknown since some cities are located entirely within counties that are already being considered only as a whole unit. Figure 1 displays their map showing the units that are immutable in their redistricting simulation. By imposing the strong

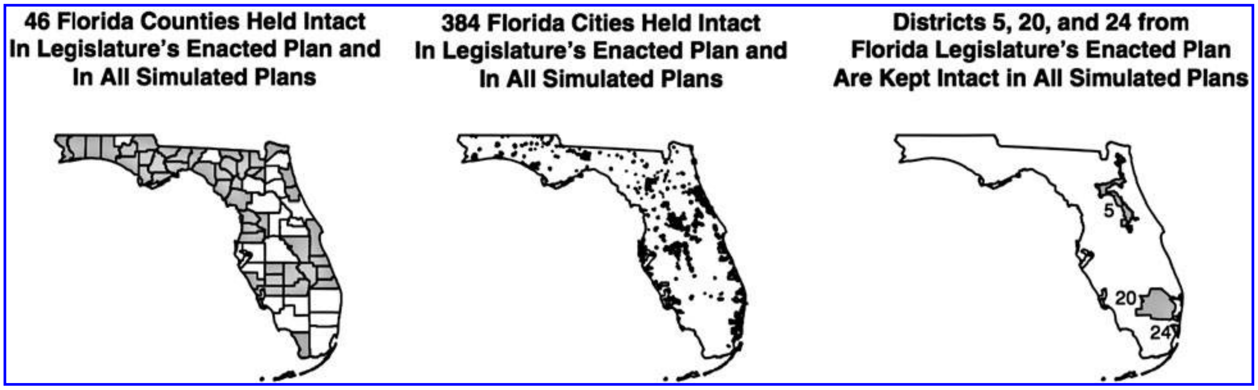

FIG. 1. Chen and Rodden's Florida simulation data.

constraint that so much of the state's geography remain single aggregated units, they produce a much more manageable computational simulation but also severely limit the interpretation of their results.

Importantly and legally consequential, the substantive justification they provide for aggregating small units into larger immutable units limits their computationally simplified approach. When we are attempting to understand or judge a disputed plan and whether it is biased or unfair, giving "deference" to a plan in dispute in this way is substantively questionable. While it makes sense to conduct a simulation where the resulting plans respect political subdivisions at least as much as the plan in dispute, requiring that every simulated map holds the *exact* same cities, counties, and districts intact is clearly overly restrictive and has substantive and biasing implications. Holding entire districts together from a disputed plan is an even more questionable choice when the objective is to analyze the fairness of a plan. These types of decisions make the computation feasible but obscure and regulate what their simulations then represent. Since it is not clear that their simulation method would even be possible without these simplifications, the Chen and Rodden approach remains significantly limited and realistic in only stylized terms.

We also embrace the idea of using simulation for redistricting analysis. Much like how using a ruler for measurement is not our idea, we also claim no credit for proposing that statistical and computational methods are useful for understanding redistricting. The fields of computer science and statistics surely provide an embarrassment of useful raw materials. Crafting and tailoring a useful tool from these fields that will unfold greater capability and capacity to analyze gerrymandering cases, however, is a non-trivial undertaking that requires deep domain knowledge, ingenuity, and careful thought. In this quest, we have begun by arguing for a more nuanced understanding of the proper baseline comparison set. By focusing on the "reasonably imperfect set" of redistricting plans, we are able to make substantial progress on both the substantive fit and the computational progress in developing appropriate redistricting tools. We avoid making simplifications to the problem for the sake of computational tractability. Instead, our method is both much more closely tailored to the substantive problem and more computationally feasible. Additionally, we devote significant effort toward the efficiency of the computational algorithm, which enables us to produce not only *orders of magnitude* more plans in the comparison set at lower levels of geographic granularity than ever before, but also an independent set of plans that meets specified goodness criteria. We present not just another simulation, but a computational model that is far more flexible, adaptable, and reflective of the actual redistricting process than any previous proposal. Our tool is novel and enables the type of analysis that is behooved by First and Fourteenth Amendment challenges to partisan gerrymanders.

THE TOOL VERSUS THE STANDARD

To be clear, on an important point, our measurement tool does not obviate the hard choices that the courts must make in order to establish a legal standard that might incorporate these tools. The tool is necessary but not sufficient since its use is neither fixed nor completely obvious. If we use a ruler to measure the length of an ant, it is clear that what the ruler provides is a measurement of the length; this measurement from the ruler is not accompanied with a judgment about the length. We may be able to say that the ant is 0.6 inches, but whether an ant is "too short" or "too long" is a separate question. The measurement is required to make a judgment, but the two are plainly not equivalent. Measurement does not imply judgment. Determining whether any plan is so unfair as to be unconstitutional or undesirable requires making legal judgments about what constitutes excess. This requires a standard that distinguishes a "reasonably imperfect redistricting," from one that is "excessively unfair." We have recommendations in this regard but are well aware that the ultimate determination must be made by the Court. In *LULAC*, Justice Stevens states that it is in the realm of the Court to make normative judgments about whether the line has been crossed in partisan gerrymandering. He comments that "Justice Kennedy faults proponents of the symmetry standard for 'not providing a standard for deciding how much partisan bias is too much.' But it is this Court, not proponents of the symmetry standard, that has the judicial obligation to answer the question of how much unfairness is too much."[8]

One manner in which to think about the way to use the tool is to begin with the notion that measures of unfairness are only able to produce values that lie in a range. In addition to developing appropriate measures of unfairness, we also need a way to think about what then defines fair and unfair, which are dichotomous choices. Perhaps an unfair plan produces a bias in excess of some number, or the unfairness measure lies outside some specified range, such as 95% of all fully balanced, simulated, reasonably imperfect plans. This step is critical but not simple.[9] The idea of establishing a cutoff based on data is a viable direction for the Court if the tools exist for the Court to assess how to determine the number that would satisfy rigorous legal standards and reasoning.

A COMPUTATIONAL APPROACH

Statistical approaches with limited data and computation have been tremendously insightful on any number of realms. More recently, with the proliferation of significant computing power, we have discovered the extensive and often surprising reach of technology, information, and computation into many realms of life. These very same capacities can shed insight into our governance structures, ideally enabling us to improve our democratic society. This is our approach here—to integrate technological advances with our articulated strategy for analyzing, contextualizing, and understanding redistricting plans. Our approach is unique and computationally intensive but also theoretically attractive.

Possible districting maps

Drawing electoral maps amounts to arranging a finite number of indivisible geographic units into a smaller number of larger areas. For simplicity, call the former "units" and the latter "districts." Since every unit must belong to exactly one district, a map is a partition of the set of all units into a pre-established number of non-empty districts. The redistricting problem is an application of the set-partitioning problem that is known to be *NP*-complete and computationally challenging (Garey and Johnson 1979). The total number of possible maps when drawing k districts using n units is a Stirling number of the second kind, $S(n, k)$ (Keane 1975), defined, combinatorially, as the number of partitions of an n-element set into k blocks, which is why it equals the number of partitions of n units into k districts. The Stirling number of the second kind can be computed recursively as $S(n, k) = k\, S(n-1, k) + S(n-1, k-1)$, which is valid when $n \geq 1$ and $1 \leq k \leq n$. Even with a modest number of units, the scale of the unconstrained map-making problem is awesome. If one wanted to divide $n = 55$ units into $k = 6$ districts, the number of possibilities is 8.7×10^{39}, a formidable number. There have been fewer than 10^{18} seconds since the beginning of the universe. Of course, the number becomes significantly smaller with relevant legal constraints, such as contiguity, in place. However, it does not become manageably smaller. The problem remains massive.

[8] 126 S. Ct. at 2638 n.9.

[9] In *Whitmore*, the proponents propose a 7% standard. However, without a fully balanced comparison set, it is not possible to understand or justify this 7% figure. That number was, moreover, derived from historical data with populations that differ from the one under consideration, which we have already discussed as problematic.

Formalization of the redistricting problem

We can create a computational model that will identify plans as it attempts to optimize some objective (e.g., competitiveness, safe districts, incumbent protection) while simultaneously satisfying legal constraints (e.g., contiguity and equipopulous districts). In this framework, there are many ways to specify an objective function. One specific formulation of the problem might proceed as follows. We have a set of N geographic units, u_1, u_2, …, u_N, that we wish to partition into a set of K districts, $d_1, d_2, \ldots, d_K$. We can create an $N \times N$ adjacency matrix, $\mathbf{C}$, to indicate the contiguity of the various units, where the entries are defined as

$$c_{ij} = \begin{cases} 1 & \text{if unit i and unit j are adjacent or i = j} \\ 0 & \text{otherwise.} \end{cases}$$

for $1 \leq i \leq N$ and $1 \leq j \leq N$. The convention that $c_{ij} = 1$ for $i = j$ is adopted to simplify the checking of connectedness of districts. The population of the N units is denoted by $p_1, p_2, \ldots, p_N$. So, if the districts are equipopulous, then the population in each district would be the average population, P, given by

$$P = \frac{1}{K} \sum_{i=1}^{N} \text{p}i.$$

Let $\mathbf{X}$ be an $N \times K$ matrix with elements, x_{ik}, denoting our decision variables. To specify a map, these variables are chosen for $1 \leq i \leq N$ and $1 \leq k \leq K$ so that

$$x_{ik} = \begin{cases} 1 & \text{if unit } u_i \text{ is assigned to district } d_k \\ 0 & \text{otherwise.} \end{cases}$$

The population in district k is then

$$P_k = \sum_{i=1}^{N} x_{ik}\, p_i \quad \text{for } k = 1,\ 2, \ldots, K.$$

We have constraints of at least three types.

1. Each unit must be assigned to exactly one district,

$$\sum_{k=1}^{K} x_{ik} = 1 \quad \text{for } i = 1,\ 2,\ \ldots, N.$$

2. The maximum population deviation across all K districts is no greater than a specified value M. For any two districts, d_i and d_j,

$$|P_{d_i} - P_{d_j}| \leq M \qquad \text{for } i,\ j = 1,\ 2, \ldots, K.$$

3. The units in each district must form a connected set. That is, each unit is accessible from any other in the set via transitions encoded in the adjacency matrix C.

Subject to the constraints above, we seek to optimize a specific objective function. The particular specification of the objective function is flexible, adaptable to any substantive interest.

We may formulate a population criteria as

$$p = \frac{\max_k(P_k) - \min_k(P_k)}{\sum_{k=1}^{K} P_k}. \tag{1}$$

This measures the level of population deviation between the set of K districts. When the districts have identical population, $p = 0$. As their population increasingly deviates from one another, the value of p increases. In this formulation, it is possible for the value of p to exceed 1. This occurs when the difference in population between districts is very large. In these cases, we can set p to its maximum value, 1, which already represents an extreme population difference where one district has more than twice the population of another.

Compactness may also be specified in many different ways (Li, Goodchild, and Church 2013). Area–perimeter criteria are popular and were first proposed by Ritter in 1882 (Frolov 1975). These measures compare the perimeter of a shape to the area of the shape. With these measures, a circle is the most compact shape and would have an area–perimeter ratio or compactness value of 1. The value of a simple area–perimeter ratio would vary with the size of the shape, but we can create a scale invariance by dividing the area by the square of the perimeter. We can also standardize the measure to have values in the [0, 1] range by including π in the numerator. This measure is the most widely used compactness measure in the area–perimeter class of measures, and the one we use, C_{IPQ}, (Osserman 1978), defined as

$$C_{IPQ} = \frac{4\pi A}{P^2}. \tag{2}$$

In the literature, this has also been called the Polsby-Popper compactness criterion (Polsby and Popper 1991).

The population formulation and the compactness criterion are flexible. There are many ways in which they may be specified. One nice feature of our particular specifications is that the values can be standardized so that they span the same [0, 1] range with a common best value. When the criteria are standardized in the same range, it simplifies the specification of weights in a multi-objective function that combines several measures reflecting competing interests that are ideally and simultaneously satisfied. Once measures of the individual criteria are specified, the user may deploy any desired customized notion of how various criteria should be weighted. These specifications are modular, flexible, and customizable across a wide set of preferences, interests, and constraints.

We have extended significant effort toward creating an efficient and substantively appropriate algorithm. We provide only an overview here since the algorithmic details are available in our published work in the operations research realm (Liu, Cho, and Wang 2016). Here, we focus on the important explication of how to utilize such a tool to gain insight into political gerrymandering. As a short summary, our computational approach is based in strong substantive knowledge and deep adherence to Supreme Court mandates. Since the spatial configuration of the geographic units plays a critical role in the effectiveness and numerical efficiency of redistricting algorithms, we have designed spatial evolutionary algorithm operators that incorporate spatial characteristics to efficiently and effectively search the solution space. Our parallelization of the algorithm at an immense scale further harnesses massive parallel computing power provided by supercomputers via the coupling of evolutionary search processes and a highly scalable message-passing model that maximizes the overlapping of computing and communication at runtime. Our experimental results demonstrate the desirable effectiveness and scalability of our approach (up to 131,072 processors) for solving large redistricting problems, which enables finely tailoring the algorithm to the substantive requirements of redistricting applications.

MARYLAND

We now turn to an illustration of how our method sheds insight into the partisan gerrymandering debate with data from Maryland. In *Shapiro v. McManus*, a bipartisan group of Maryland voters alleged that the 2011 congressional districts constitute a partisan gerrymander that violates their First Amendment right to participate in electing their political leaders. The claim is that the state has drawn the districts in such a way that it infringes upon a partisan voter's ability to elect his or her candidate of choice. To be sure, all plans have some partisan effect. This is unavoidable even in a completely partisan-free process. Moreover, even when the line drawers do not inject excessive partisan biases, some difference in partisan effect may occur simply as a byproduct of the peculiarities in the population landscape or from trying to achieve other important state goals such as equal population, compactness, and preserving communities of interest. At the same time, it is possible that partisan considerations were overwhelming or the predominant motivation behind how the district lines are drawn. The essence of deciding these cases, then, is to *determine how one might be able to untangle the underlying motives using the data alone*. In order for the Court to make judgments about the role of partisanship in drawing district lines, it needs some way to quantify the role of partisanship in the line drawing process.

An important consideration here is how one might construct a counterfactual for analysis. We have an electoral map that satisfies a set of legal and non-partisan criteria. This always includes population equality and may include, depending on the jurisdiction, criteria such as compactness and respect for political subdivisions. We know that the line drawers sought to satisfy some set of non-partisan criteria but are unsure whether or to what extent partisan considerations may have also come into play. Given this scenario, *the counterfactual set of maps is the set of plans that are at least as good or better on non-partisan factors because these are known considerations, but do not consider partisanship*. If the line drawers did not consider partisanship, then their plan should resemble the broad contours of this counterfactual set. The important gain from the computation is that while it is difficult to objectively ascertain how much a human considered partisanship in the drawing of lines, it is not difficult to ensure that partisanship is not a consideration in a computer-drawn map because it is simple to either specify and/or verify that partisan data not be used at all or to set the level at which partisanship will be considered in relation to other non-partisan factors. For a partisan gerrymandering case, having a set of plans that are drawn without partisan considerations but exhibit comparable non-partisan metrics allows us to see how the alleged partisan considerations in the disputed plan substantively alter the outcomes that emerge from a less or non-partisan process.

To engage in creating this baseline reasonably imperfect comparison set, we begin by taking measurements from the disputed plan. For Maryland, on two non-partisan measures, population equality and compactness, we required that a reasonably imperfect plan have non-partisan metrics that are at least as good as the disputed plan. We deem these "goodness thresholds." Certainly, additional non-partisan criteria may be included here. These non-partisan factors should be chosen to match the peculiarities of the legal case. In our algorithm, as long as a particular facet can be quantified, we can incorporate it into our algorithm. We did not incorporate any partisan information into our map-drawing process.

For the state of Maryland, our algorithm identified more than a billion "legal maps." Of this set, about 250 million of these maps satisfy the requirements of reasonably good and feasible maps that would fit in the baseline set (i.e., they are as good as the current map on non-partisan criteria). From the 250 million maps, we retained only one feasible map every time 1,000 maps were generated. This is an additional way to inject independence among the set of maps, leaving us 258,584 maps for our final comparison set. While others may have attempted to simulate maps, none of those attempts have even remotely approached either the quantity or quality of the maps that we are able to identify with our tool. We have created *orders of magnitude* more maps than ever previously achieved. This is a significant achievement and a testament to the efficiency of our search algorithm. In addition, there are not simply "more maps." Our search criteria identifies maps over a goodness threshold. They are not just "random maps" devoid of desirable characteristics and highly unlikely to be considered as a serious map that might become law, but maps that are "reasonably imperfect" in the sense that they are at least as good as the current map on non-political measures.

The partisan characteristics of plans drawn without partisan information

There are a number of ways to measure the partisan nature of an electoral map. Our algorithm is flexible and agnostic about what measure might capture partisan leaning best; any measure that can be quantified can be computed and examined. Two of these measures, responsiveness and bias, derive from the seats–votes curve (Edgeworth 1898; Taagepera 1973). To gain a sense of these measures, consider a generic seats-votes curve shown in Figure 2. The solid diagonal line represents a proportional representation system where every gain of x in vote proportion translates into exactly a gain of x in the seats proportion. On the other hand, the dashed S-shaped curve depicts a different relationship between vote proportion and seat proportion. For the S-shaped relationship, between vote proportions of approximately 0.40 and 0.60, the seat gain is most responsive to changes in vote proportion. In contrast, near 0.0 and 1.0, the system is not very responsive.

Responsiveness is a measure of how sensitive seat gains are to changes in vote proportion. In the seats–votes curve, responsiveness can be seen simply as the steepness in the slope of the curve. The steeper the slope, the more responsive the seat proportion (or the faster the change in seat gain responds) to changes in vote proportion. The S-shaped curve has a steep slope near the middle but flattens out considerably toward the extreme vote proportions of 0 and 1. Responsiveness, $R(V)$ can thus be defined as

$$R(V) = \frac{dE(S \mid V)}{dV}. \tag{3}$$

The larger the value of $R(V)$ is, the more responsive the districting plan is to vote proportion and thus the weaker the argument for a gerrymander.

The plot on the left in Figure 3 shows how the responsiveness of the current electoral map compares with the responsiveness of the 258,584 maps that were retained as our baseline comparison set.[10] As we can see from the plot, in Maryland, there are many maps that are legally viable, at least as good as the current map on chosen non-partisan indicators, but are yet more responsive to the vote. In fact, of the set of reasonably imperfect maps our algorithm identified without using any partisan data whatsoever, 94.79% of the generated maps were

[10]For each of these plots, data from six elections are incorporated. These six elections (U.S. House, U.S. Senate, State House, Governor, Secretary of State, and Auditor) all involved the same electorate and the current districting scheme. These were the data available to us, but any set of elections for which one has data can be used.

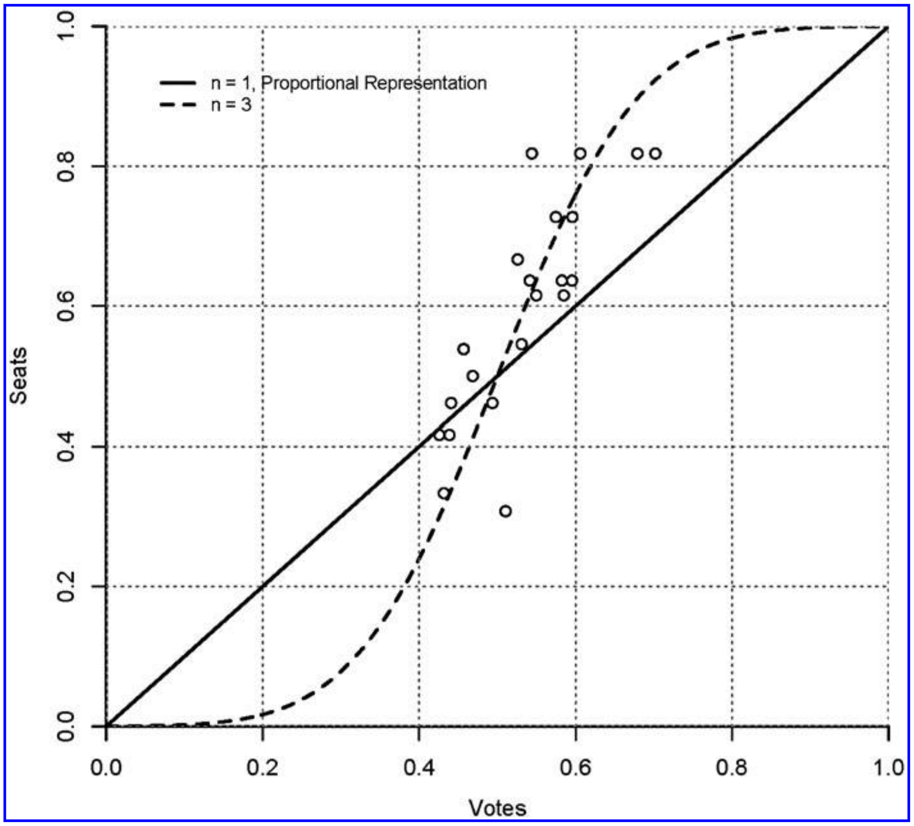

FIG. 2. Seats-votes curves.

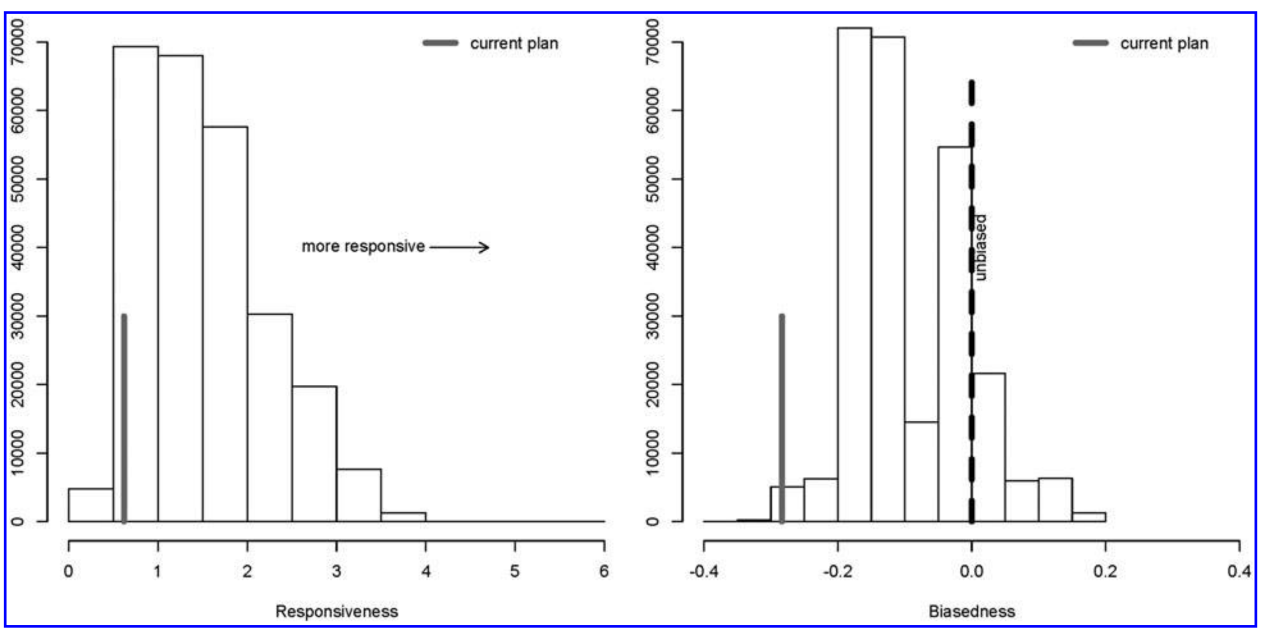

FIG. 3. Current plan versus 258,584 reasonably imperfect plans on responsiveness and bias.

more responsive to changes in the vote proportion than the current map. This implies that partisan considerations were likely at play in devising the current map since creating a map with its level of responsiveness is unusual (though possible) when partisanship is not a factor in the map creation.[11]

Biasedness is the condition of favoring one party over the other and can be described as a deviation from bipartisan symmetry. Based on the seats–votes curve, biasedness, $B(V)$, can be defined as

$$B(V) = E(S \mid V) - [1 - E(S \mid (1-V))]. \quad (4)$$

In other words, both parties should expect to receive the same number of seats given the same vote proportion. If one party receives more seats than the other given the same vote proportion, then there is a lack of symmetry or a bias toward the party that would gain more seats under the same vote proportions or conditions. Here, the ideal symmetry is achieved at 0 while values that are negative show a bias toward the Democrats and positive values indicate a skew toward the Republicans. Measures of biasedness are shown in the plot on the right in Figure 3. Like the responsiveness plot to its left, the biasedness of the current Maryland map falls on the left end of the histogram. The current map is quite skewed in its favor toward the Democrats. Among the maps in the baseline set, the current map is more extreme than 99.79% of the generated maps on the biasedness measure. In Maryland, it appears that while the population landscape tends toward maps that seem to virtually always favor Democrats, the current plan has about as strong a Democratic bias as any plan we could identify with comparable non-partisan characteristics, again implying that partisan information was used in building the current map.

There are also other ways to measure the influence of partisan motivations besides those related to the seats–votes curve. For instance, one might imagine that the most competitive (and therefore non-partisan) map would have districts that are evenly divided between Democratic and Republican registrants and that any advantage toward one party or the other would be symmetrically split. That is, if there are eight districts, then if four districts leaned slightly Democratic, then the other four would lean slightly Republican. One way to quantify this measure of competitiveness is as follows. Let D_k be the Democratic registration in district k and R_k be the Republican registration in district k for $k = 1, 2, \ldots, K$. A district is most competitive when $D_k = R_k$. Across all districts, an overall measure of competitiveness could be calculated as

$$f = T_p \, (1 + \alpha T_e) \, \beta, \quad (5)$$

where

$$T_p = \frac{1}{K}\left(\sum_{k=1}^{K}\left|\frac{R_k}{D_k + R_k} - \frac{1}{2}\right|\right),$$

$$T_e = \left|\frac{B_R}{K} - \frac{1}{2}\right|, \quad \text{for } 0 \le T_p \le 0.5, \; 0 \le T_e \le 0.5, \text{ and } \alpha, \beta > 0.$$

Here, T_p measures competitiveness as a deviation of the Republican two-party registration from 0.50 in each district and T_e is a weighting factor, which captures the differential in the number of seats won by the two parties. In the formulation for T_e, B_R is the number of districts where Republican registration is larger than the Democratic registration; α defines the weight of T_e in the competitiveness measure; and β is a normalizing constant so that the value of f spans the range [0, 1]. For example, since $T_e \in [0, 0.5]$ when $\alpha = 1$, if we set $\beta = \frac{4}{3}$ then $f \in [0, 1]$. This formulation is small when the average registration differential, T_p, is small while also ensuring that this small differential is spread equally between the parties. When $f = 0$, Republican registration and Democratic registration is the same and the number of districts where Republicans dominate and the number of districts where the Democrats dominate is identical.

The histogram on the left in Figure 4 shows how the electoral maps fall on competitiveness. It appears that the geographic landscape and the constraints on compactness and population equality constrain competitiveness to some extent. However, without considering partisanship, the large proportion of our maps were more competitive than the current Maryland map, which was more competitive than only 12.44% of our generated maps. It appears likely, then, that the line drawers used information

[11] We do not intend to forward concrete tests or cutoffs for the Court, but we note here that if one were using a strict 5% cutoff, which is common in many statistical analyses, this map would just slide by, satisfying what one might call 0.05-level significance but failing to pass a significance test at the 0.053-level.

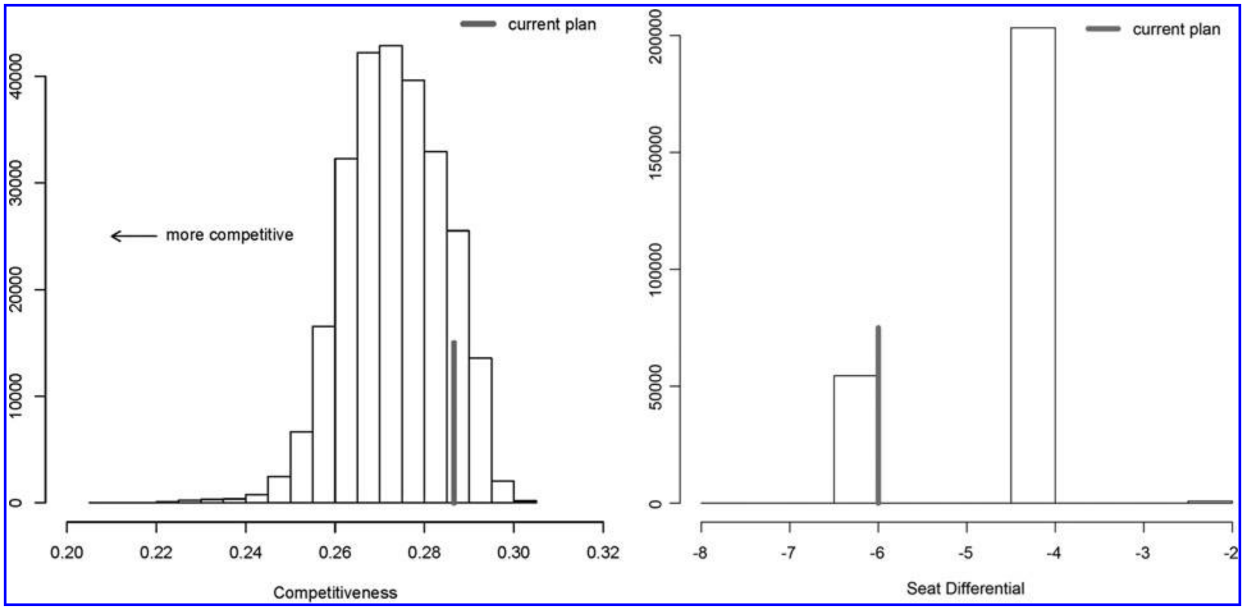

FIG. 4. Current plan versus 258,584 reasonably imperfect plans on competitiveness and seat differential.

that results in the maps being less competitive than they would be if no partisan information was considered.

In the right plot in Figure 4, we see the seat differential between the Democrats and the Republicans. In the vast majority (78.61%) of our generated maps, the Democrats have a four-seat advantage. That is, the Democrats have an advantage in six seats while the Republicans have an advantage in two seats. In a smaller set (21.07%), they have a six-seat advantage (as they do in the current map with seven Democratic seats and one Republican seat). In a few plans (824 or 0.32%), the generated map is as good as the current map on non-partisan factors and has only a two-seat differential (or Democrats with five seats and Republicans with three seats). In eight plans (< 0.0002%), the Democrats have the advantage in all eight seats while the map is as good as the current map on non-partisan factors, even though no partisanship information was used in the construction of the maps.

Since partisanship was not explicitly considered in the creation of our maps, we can regard the set of maps that we create as representative of the types of maps that are generated with no partisan intent but are still constrained by Maryland's natural population landscape, e.g., where the rivers and ocean flow, how the mountains carve up the state, how the cities have developed, the shape of its counties, the racial and/or socio-economic concentrations that have formed over the course of the state's history, and constraints on population equality and compactness. Some of these maps will appear highly partisan, but our concern is not the extremes of our distribution. Instead, we look to the first two moments of our distribution, the mean and the variance, to quantify the levels of partisan effects that are not excessive for the landscape under consideration. Accordingly, when we notice these levels of partisan effects that are identified by our analysis, the Court should not regard them as excessive or in need of regulation. On the other hand, if the disputed plan registers partisan levels at the extremes (where extreme is defined by the Court) of the baseline set, then there may be cause for concern.

The distributions allow us to assess whether the existing plans are outliers among other reasonably imperfect plans that could have been drawn, allowing us to understand whether, among the possibilities, is this proposed plan particularly unresponsive to voter preferences? Is this proposed plan exceptionally biased toward one party? Could we have achieved the goals of this new plan while maintaining greater respect for other important criteria or traditional districting principles such as respect for political subdivisions or compactness? Is the shift toward a Republican or Democratic bias a function of shifting demographics and population migration, or are the motivations of the partisan line drawers the driving force? If the proposed plan is not exceptional in any way but is still biased toward one party, then the Court may decide that the grounds do not exist for revisiting the proposed plan. The pivot lies not within the plan itself or simulations based on one particular plan but in how that plan compares to other possibilities. In this way, the ability to generate and analyze a large number of feasible redistricting plans without making a host of simplifications for computational reasons is essential for ensuring our democratic values.

Our plans still exhibit partisan effects as all maps do, but inarguably, our maps are not *intentionally* biased and have no underlying partisan motivation. Drawing maps in this way allows us to separate how the population patterns in the state constrain the map making and how the partisan motivations might alter the creation of maps. If partisanship is not considered at all, it appears that the modal plan is more responsive, less biased, more competitive, and would give Democrats an advantage in six seats and the Republicans an advantage in two seats. In terms of responsiveness, the map under contention is particularly unresponsive to voters. It is also particularly biased. These histograms provide evidence that under a First Amendment framework, the map has encroached one party in favor of the other, and that these infringements were the result of an explicit consideration of party, not necessitated by the population landscape. Whether this impingement is excessive is left to the Court. Our analysis also shows that there are a slew of map alternatives that would significantly remedy the disputed plan's partisan effects while maintaining respect for the non-partisan criteria. Our analysis here is illustrative of what can be achieved. In an actual legal case, the lawyers or the judge may wish to expand the non-partisan constraints to include, for instance, respect for particular political subdivisions. We did not include that here, but this is a simple extension. Different partisan measures, such as the efficiency gap, may also be desired. Each legal case is idiosyncratic. Our model is easily pliable to these types of particulars.

We also acknowledge that our results are limited because we did not incorporate a racial measure. We omitted this analysis for three reasons. First, it is unclear how racial and partisan considerations should interact. The Court has not been clear on this issue except to say that there should be some balance. Second, while our tool can be used to examine racial gerrymandering as well, that is the subject of separate but related research. Our particular implementation for racial considerations is nuanced and tailored toward the extensive legal discourse on racial gerrymandering, which is far more extensive than the partisan gerrymandering discourse. Racial and partisan considerations can be combined in a variety of ways, but we do not wish to push a particular view on that matter given the Court's vagueness on the subject. It should be unambiguous, however, that the tool we propose is immensely malleable, flexible, highly adaptable, and based on straightforward logic. Once the Court clarifies these matters, it is not difficult to mold the use of the tool to the Court's declarations. Finally, we are not intending to submit a policy recommendation. We wish simply to discuss and demonstrate how our tool is useful for assessing partisan gerrymandering. Our work in this realm overcomes many significant limitations that have plagued this exercise in the past. We can now produce maps that closely mimic the redistricting process with existing computing technology.

DISCUSSION

We have lacked sufficient tools to analyze redistricting data, in part, because the computing problem is massive. While the frameworks proffered in the literature are theoretically attractive, the tools necessary to implement these methods have been elusive. However, while half a century has passed since we began discussing the theoretical attractiveness of simulations, we have not updated our tool set to be more tightly tailored to the precise contours of the Court's mandates. Even recent attempts at simulations have significantly simplified the problem for computational tractability. Surely it is not simple or straightforward to devise these tools, but the time is ripe for us to think harder about how to develop the correct baselines for analysis and to be retrospective about the limitations of any approach. While we are not the first to propose the use of a baseline or the use of simulations, our simulation design is novel and more meticulously adherent to the realities of redistricting cases. To this end, we also create and demonstrate a far superior tool that creates a much closer rendition of the baseline that is necessary to make legal challenges viable.

It is difficult to foresee how the future will unfold. At the same time, it seems clear that we are in the midst of an unmistakable and enormous technological transformation. The recent proliferation of computing power has led to the discovery of the extensive and often surprising reach of technology, information, and computation in many realms of life. Our capacities to compile, organize, analyze, and disseminate information have increased dramatically and facilitated the creation of many tools to connect citizens and automate human tasks. We are now poised at the cusp of being able to use statistical modeling and computing technology in the redistricting process in an unprecedented manner, allowing us to understand redistricting in fundamentally new ways. Visions of the future often entail new modes of transportation and greater automation of basic tasks, but we must also think and consider how to improve society by integrating technological advances with our articulated democratic theories and ideals.

REFERENCES

Bishop, Bill. 2008. *The Big Sort: Why the Clustering of Like-Minded America is Tearing Us Apart*. New York: Houghton Mifflin.

Cain, Bruce E. 1985. "Simple v. Complex Criteria for Partisan Gerrymandering: A Comment on Niemi and Grofman." *UCLA Law Review* 33: 213–226.

Cain, Bruce E. 2012. "Redistricting Commissions: A Better Political Buffer?" *Yale Law Journal* 121(7): 1808–1844.

Chen, Jowei and Jonathan Rodden. 2015. "Cutting through the Thicket: Redistricting Simulations and the Detection of Partisan Gerrymanders." *Election Law Journal* 14(4): 331–345.

Edgeworth, F. Y. 1898. "Miscellaneous Applications of the Calculus of Probabilities." *Journal of the Royal Statistical Society* 61(3): 534–544.

Frolov, Y. S. 1975. "Measuring Shape of Geographical Phenomena: A History of the Issue." *Soviet Geography Review and Translation* 16(10): 676–687.

Garey, M. R. and D. S. Johnson. 1979. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. New York: W.H. Freeman and Co.

Garfinkel, R. S. and G. L. Nemhauser. 1970. "Optimal Political Districting by Implicit Enumeration Techniques." *Management Science* 16(8): B-495–B-508.

Grofman, Bernard. 1985. "Criteria for Districting: A Social Science Perspective." *UCLA Law Review* 33: 77–84.

Grofman, Bernard and Gary King. 2007. "The Future of Partisan Symmetry as a Judicial Test for Partisan Gerrymandering after LULAC v. Perry." *Election Law Journal* 6(1): 2–35.

Gudgin, G. and P. J. Taylor. 1979. *Seats, Votes and the Spatial Organization of Elections*. London: Pion.

Harris, Curtis. 1964. "A Scientific Method of Districting." *Behavioral Science* 9: 219–225.

Keane, M. 1975. "The Size of the Region-Building Problem." *Environment and Planning A* 7: 575–577.

Li, Wenwen, Michael F. Goodchild, and Richard Church. 2013. "An Efficient Measure of Compactness for Two-Dimensional Shapes and its Application in Regionalization Problems." *International Journal of Geographical Information Science* 27(6): 1227–1250.

Liu, Yan Y., Wendy K. Tam Cho, and Shaowen Wang. 2016. "PEAR: A Massively Parallel Evolutionary Computational Approach for Political Redistricting Optimization and Analysis." *Swarm and Evolutionary Computation* 30 (October): 78–92.

Lowenstein, Daniel H. and Jonathan Steinberg. 1985. "The Quest for Legislative Districting in the Public Interest: Elusive or Illusory." *UCLA Law Review* 33(1):,74–75.

McGhee, Eric. 2014. "Measuring Partisan Bias in Single-Member District Electoral Systems." *Legislative Studies Quarterly* 39(1):,55–85.

Moore, Gordon E. 1965. "Cramming More Components onto Integrated Circuits." *Electronics* 38(8):,114–117.

Nagel, Stuart S. 1965. "Simplified Bipartisan Computer Redistricting." *Stanford Law Review* 17(5): 863–899.

Niemi, Richard G. 1985. "The Relationship between Votes and Seats: The Ultimate Questions in Political Gerrymandering." *UCLA Law Review* 33: 185–212.

Niemi, Richard G., Bernard Grofman, Carl Carlucci, and Thomas Hofeller. 1990. "Measuring Compactness and the Role of a Compactness Standard in a Test for Partisan and Racial Gerrymandering." *Journal of Politics* 52(4): 1155–1181.

Niemi, Richard G. and John Deegan. 1978. "A Theory of Political Districting." *American Political Science Review* 72(4): 1304–1323.

Niemi, Richard G. and Simon Jackman. 1991. "Bias and Responsiveness in State Legislative Districting." *Legislative Studies Quarterly* 16(2): 183–202.

Osserman, Robert. 1978. "Isoperimetric Inequality." *Bulletin of the American Mathematical Society* 84(6): 1182–1238.

Papayanopoulos, L. 1973. "Quantitative Principles Underlying Apportionment Methods." *Annals of the New York Academy of Sciences* 219: 3–4.

Polsby, D. and R. Popper. 1991. "The Third Criterion: Compactness as a Procedural Safeguard Against Partisan Gerrymandering." *Yale Law and Policy Review* 9(2): 301–353.

Schwartzberg, Joseph E. 1965. "Reapportionment, Gerrymanders, and the Notion of 'Compactness.'" *Minnesota Law Review* 50: 443–452.

Shepherd, J. W. and M. A. Jenkins. 1970. "Decentralizing High School Administration in De- troit: A Computer Evaluation of Alternative Strategies of Political Control." *Economic Geography* 48(1): 95–106.

Stephanopoulos, Nicholas O. and Eric M. McGhee. 2015. "Partisan Gerrymandering and the Efficiency Gap." *University of Chicago Law Review* 82: 831.

Taagepera, Rein. 1973. "Seats and Votes: A Generalization of the Cube Law of Elections." *Social Science Research* 2(3): 257–275.

Thoreson, James and John Liittschwager. 1967. "Computers in Behavioral Science: Legislative Districting by Computer Simulation." *Behavioral Science* 12: 237–247.

Vickrey, William S. 1961. "On the Prevention of Gerrymandering." *Political Science Quarterly* 76: 105–110.

Weaver, J. B. and S. W. Hess. 1963. "A Procedure for Nonpartisan Districting: Development of Computer Techniques." *Yale Law Journal* 72: 288–308.

Address correspondence to:
Wendy K. Tam Cho
Department of Political Science
University of Illinois at Urbana-Champaign
420 David Kinley Hall
1407 W. Gregory Drive
Urbana, IL 61801

E-mail: wendycho@illinois.edu